CHANGE OF VARIABLES IN THE LAPLACE TRANSFORM AND SOME APPLICATIONS

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The effect of a change of variable by an arbitrary nonlinear function in the Laplace transform and the inverse transform is considered. Some applications of the results to heat-conduction problems are discussed.

The properties of the Laplace transform under a linear transformation of the argument in the original function or transform are generally well known. In the present paper, the properties of the Laplace transform under a nonlinear transformation of the argument are discussed, continuing a study begun in [1]. As is well known, finding the inverse transform is the most difficult step in the operational method. The properties discussed here should widen significantly the class of functions whose inverse transforms can be found.

Change of Variables in the Transform Function

It was shown in [1] that the replacement of the transform variable p by an arbitrary function K(p) in the Laplace transform $U_1(p)$ leads to the following result for the inverse transform:

$$u(t) = L^{-1}U_1[K(p)] = \int_0^\infty u_1(\tau)\,k(t,\ \tau)\,d\tau; \tag{1}$$

$$k(t,\ \tau) = L^{-1}\exp[-K(p)\tau]. \tag{1a}$$

In (1) and (1a) the analytic function K(p) appearing in the transform U_1 must satisfy the condition of "transformability," i.e., we must have $\mathrm{Re}[K(p)] > \sigma_1$ for $\mathrm{Re}(p) > \sigma_1$ where σ_1 is the exponent of growth of $u_1(t)$. A simple form which satisfies this condition is $K(p) \to \beta_0 p^\alpha$ as $p \to \infty$, where $-1 \leqslant \alpha \leqslant 1$ and $\beta_0 > 0$. Using (1) and the properties of the Laplace transform we can obtain some useful results.

1. Replacement of the Variable by a Sum of Functions. If $K(p) = K_1(p) + K_2(p)$ and both functions satisfy the condition of transformability then [1]

$$k(t,\ \tau) = \int_0^t k_1(\xi,\ \tau)\,k_2(t-\xi,\ \tau)\,d\xi; \tag{2}$$

$$k_i(t,\ \tau) = L^{-1}\exp[-K_i(p)\,\tau]. \tag{2a}$$

In the special case where $K_1(p) = d_1p + d_0$ (a linear function) and $K_2(p)$ is nonlinear, we get

$$k(t,\ \tau) = \mathrm{H}(t-d_1\tau)\exp(-d_0\tau)\,k_2(t-d_1\tau,\ \tau),\ \ d_1 > 0; \tag{3}$$

$$u(t) = L^{-1}U_1[d_1p + d_0 + K_2(p)] = \tag{3a}$$

$$= \int_0^{t/d_1} \exp(-d_0\tau)\,k_2(t-d_1\tau,\tau)\,u_1(\tau)\,d\tau = \frac{1}{d_1}\int_0^t \exp\left(-\frac{d_0}{d_1}\tau\right)k_2\left(t-\tau,\ \frac{\tau}{d_1}\right)u_1\left(\frac{\tau}{d_1}\right)d\tau,$$

where H(t) is the Heaviside unit step function. Equation (2) can be easily generalized to several functions.

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2. Replacement of the Variable by a Compound Function. Let $K(p) = K_1[K_2(p)]$ be a compound function where the arbitrary analytic functions $K_i(p)$ are transformable. Then (1a) takes the form

$$k(t, \tau) = \int_0^\infty k_1(\xi, \tau) k_2(t, \xi)\, d\xi, \tag{4}$$

where $k_i(t, \tau)$ is given by (2a). Indeed, applying (1) successively from right to left to the chain identity

$$U(p) = U_1[K_1[K_2(p)]] = U_2[K_2(p)],$$

where $U_2(p) = U_1[K_1(p)]$, we obtain the desired form:

$$u(t) = \int_0^\infty u_2(\xi) k_2(t, \xi)\, d\xi = \int_0^\infty \left[\int_0^\infty u_1(\tau) k_1(\xi, \tau) d\tau\right] k_2(t, \xi)\, d\xi = \int_0^\infty u_1(\tau) \left[\int_0^\infty k_1(\xi, \tau) k_2(t, \xi)\, d\xi\right] d\tau.$$

In the proof of (4) we assumed the existence of the transform $U_2(p)$ and that the orders of integration could be interchanged [this amounts to the assumption that the inside integral determining $u_2(\xi)$ is uniformly convergent with respect to ξ for $\xi \geqslant 0$]. We note that property (4) is noncommutative; for $K(p) = K_2[K_1(p)]$ Eq. (1a) takes the form

$$k(t, \tau) = \int_0^\infty k_2(\xi, \tau) k_1(t, \xi)\, d\xi. \tag{5}$$

Relations (4) and (5) can be generalized easily to the superposition of several functions. If $K(p)$ can be represented in the form

$$K(p) = K^{[n]}_{(p)} = K_n[K_{n-1}[\ldots[K_2[K_1(p)]\ldots], \quad n = 2, 3, \ldots, \tag{6}$$

then the corresponding function is $k(t, \tau) = k^{[n]}(t, \tau)$, for which it is not difficult to obtain the recurrence formula

$$k^{[n]}(t, \tau) = L^{-1}\exp[-K^{[n]}(p)\tau] = \int_0^\infty k_n(\xi, \tau) k^{[n-1]}(t, \xi)\, d\xi. \tag{6a}$$

If $K(p)$ is represented in the form

$$K(p) = \tilde{K}^{[n]}(p) = K_1[K_2[\ldots[K_{n-1}[K_n(p)]\ldots], \quad n = 2, 3, \ldots, \tag{7}$$

then the following recurrence relation applies to $k(t, \tau) = \tilde{k}^{[n]}(t, \tau)$:

$$\tilde{k}^{[n]}(t, \tau) = \int_0^\infty k_n(t, \xi) \tilde{k}^{[n-1]}(\xi, \tau)\, d\xi. \tag{7a}$$

3. Inverse Functions in the Transform. If $K(p)$ is replaces by $K^*(p)$, which is the inverse function to $K(p)$, so that $K^*[K(p)] = p$, then the inverse transform $u_1(t)$ can be expressed in terms of $u(t)$ by

$$u_1(t) = \int_0^\infty u(\tau) k^*(t, \tau) d\tau; \tag{8}$$

$$k^*(t, \tau) = L^{-1}\exp[-K^*(p)\tau]. \tag{8a}$$

Here $K^*(p)$, like $K(p)$ itself, must be transformable. Thus (1) and (8) are a pair of reciprocal integral transforms whose kernels are given by (1a) and (8a). It is not difficult to show that

$$\delta(t - \xi) = \int_0^\infty k(t, \tau) k^*(\tau, \xi)\, d\tau.$$

If the transform function is the superposition of several functions so that $K(p) = K^{[n]}(p)$, then the inverse function $K^{*[n]}(p)$ can be expressed in terms of the separate inverse functions $K_m^*(p)$ as follows:

$$K^{*[n]}(p) = K_1^* [K_2^* [...[K_{n-1}^* [K_n^* (p)] ...].$$

A recurrence formula for $K^{*[n]}(t, \tau)$ can be found using (7) and (7a):

$$k^{*[n]}(t, \tau) = \int_0^\infty k_n^* (t, \xi)\, k^{*[n-1]} (\xi, \tau)\, d\xi.$$

4. Generalized Multiplication Property of Transforms. If we are given two transforms $U_1(p)$ and $U_2(p)$ and two functions $K_1(p)$ and $K_2(p)$ which are transformable, then [1]

$$u(t) = L^{-1}\{U_1[K_1(p)]\cdot U_2[K_2(p)]\} = \int_0^t \Big[\int_0^\infty u_1(\xi)\, k_1(\tau, \xi)\, d\xi \int_0^\infty u_2(\eta)\, k_2(t-\tau, \eta)\, d\eta\Big] d\tau, \tag{9}$$

where $k_i(t, \tau)$ are given by (2a). Relation (9) can easily be generalized to the case of several transforms. If we put $K_2(p) = 1$, then it reduces to the Efros transform in a somewhat different form:

$$u(t) = L^{-1}\{U_1[K_1(p)]\cdot U_2(p)\} = \int_0^t u_2(t-\tau) \Big[\int_0^\infty u_1(\xi)\, k_1(\tau, \xi)\, d\xi\Big] d\tau. \tag{10}$$

We now consider the properties of the inverse transforms, which will turn out to be reciprocals to the results obtained for the transforms.

Change of Variables in the Inverse Transform

Upon a replacement in the inverse transform (i.e., original function), $u_1(t)$ of variable t by an arbitrary function $k(t)$ so that $u(t) = u_1[k(t)]$ the following relation holds between the transforms $U(p)$ and $U_1(p)$:

$$U(p) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} U_1(z) K(p, z)\, dz = C_z [U_1(z) K(p, z)]; \tag{11}$$

$$K(p, z) = L \exp[k(t) z] = \int_0^\infty \exp[-pt + k(t) z]\, dt. \tag{11a}$$

Here and below C_z denotes the convolution of the functions $U_1(z)$ and $K(p, z)$ with respect to the complex variable z in the right half-plane. In (11) and (11a), $u(t)$ and $u_1(t)$ are the inverse transforms, $k(t)$ does not increase faster than linearly in the limit $t \to \infty$ (i.e., $k(t) \leqslant Mt$), and it is assumed that the integral on the right-hand side of (11) is absolutely convergent in the half-plane $\mathrm{Re}(z) > \sigma_1$. In order to prove (11) one must replace $K(p, z)$ on the right-hand side by the Laplace integral and change the order of integration. As for the transforms, some useful properties can be shown to be consequences of (11) and the usual properties of the Laplace transform.

1. Replacement of the Variable by a Sum of Functions. If $k(t) = k_1(t) + k_2(t)$ and both functions increase no faster than linearly at infinity, then

$$U(p) = Lu_1[k_1(t) + k_2(t)] = C_z[U_1(z) K(p, z)]; \tag{12}$$

$$K(p, z) = C_s[K_1(s, z) K_2(p-s, z)],\ K_i(p, z) = L \exp[k_i(t) z]. \tag{12a}$$

This relation can be obtained at once from the well-known multiplicative properties of the inverse transforms. In the special case of the sum of a linear and nonlinear function $k(t) = d_1 t + d_0 + k_2(t)$

$$U(p) = C_z[U_1(z) K_2(p - d_1 z, z) \exp(d_0 z)]. \tag{13}$$

2. Replacement of the Variable by a Compound Function. Let $k(t) = k_1[k_2(t)]$ be the compound function and $k_1(t)$ increase no faster than linearly at infinity. Then (11a) takes the form

$$K(p, z) = C_s[K_1(s, z) K_2(p, s)], \tag{14}$$

where $K_1(p, z)$ is given by (12a). Applying (11) from right to left to the chain identity

$$u(t) = u_1[k_1[k_2(t)]] = u_2[k_2(t)],$$

where $u_2(t) = u_1[k_1(t)]$, and changing the order of the integration we have

$$U(p) = C_z[U_2(z) K_2(p, z)] = C_z\{K_2(p, z) C_\theta[U_1(\theta) K_1(z, \theta)]\} = C_\theta\{U_1(\theta) C_z[K_1(z, \theta) K_2(p, z)]\}.$$

3. Inverse Function in the Original. In (11) we take for $k(t)$ the function $k^*(t)$ inverse to $k(t)$ such that $k^*[k(t)] = t$ is satisfied. Then the transform $U_1(p)$ can be found in terms of $U(p)$ by the relation

$$U_1(p) = C_z[U(z) K^*(p, z)], \quad K^*(p, z) = L \exp[k^*(t) z]. \tag{15}$$

The function $k^*(t)$, like $k(t)$, must satisfy the linearity condition at infinity. Equations (11) and (15) define a pair of reciprocal integral transforms.

4. Generalized Multiplicative Property of Inverse Transforms. Let $u_1(t)$ and $u_2(t)$ be two inverse transforms and let $k_i(t)$ be a pair of arbitrary functions satisfying the linearity condition at infinity. Using the usual multiplicative properties of inverse transforms along with (11), it is not difficult to see that

$$L\{u_1[k_1(t)] u_2[k_2(t)]\} = C_s[\tilde{U}_1(s) \tilde{U}_2(p - s)]; \tag{16}$$

$$\tilde{U}_i(p) = C_z[U_i(z) K_i(p, z)], \quad i = 1, 2. \tag{16a}$$

This result is proven using the same assumptions as in (11). In the special case where $k_2(t) = t$, it reduces to the reciprocal of the Efros transformation:

$$L\{u_1[k_1(t)] u_2(t)\} = C_s[\tilde{U}_1(s) U_2(p - s)], \tag{17}$$

where $\tilde{U}_1(p)$ is given by (16a). Equation (17) can be written in the more conventional form [2]

$$L\{u_1[k_1(t)] u_2(t)\} = C_z[U_1(z) K_0(p, z)];$$

$$K_0(p, z) = L\{u_2(t) \exp[k_1(t) z]\}.$$

Examples of the use of this result in obtaining Laplace transforms and inverses are given in [2].

Some Applications

We now show how the properties discussed above can be applied to find inverse Laplace transforms. From tables of Laplace transforms [3, 4] we determine $L^{-1}\exp(-\tau h \ln p)$, $L^{-1}\exp(-\tau h\sqrt{p})$, $L^{-1}\exp(-\tau h/p)$. Using these results and (3) we can compute inverses to the following transforms:

$$u(t) = L^{-1} U_1(d_1 p + d_0 + h \ln p) = \int_0^{t/d_1} u_1(\tau) \frac{(t - d_1\tau)^{\tau h - 1}}{\Gamma(\tau h)} \exp(-d_0\tau)\, d\tau; \quad d_1, d_0, h \geqslant 0; \tag{18}$$

$$u(t) = L^{-1} U_1(d_1 p + d_0 + h\sqrt{p}) = \frac{1}{2}\sqrt{\frac{h}{\pi}} \int_0^{t/d_1} \exp(-d_0\tau) \frac{\tau u_1(\tau)}{(t - d_1\tau)^{3/2}} \exp\left[-\frac{(h\tau)^2}{4(t - d_1\tau)}\right] d\tau; \tag{19}$$

$$d_1, d_0, h \geqslant 0;$$

$$u(t)=L^{-1}U_1\left(d_1p+d_0+\frac{h}{p}\right)=\frac{1}{d_1}U_1\left(\frac{t}{d_1}\right)\exp\left(-\frac{d_0t}{d_1}\right)-$$

$$-\frac{\operatorname{sgn}(h)}{d_1}\int_0^t u_1\left(\frac{\tau}{d_1}\right)\sqrt{\frac{|h|\tau}{(t-\tau)}}\,L_1[2\sqrt{|h|\tau(t-\tau)}]\exp\left(-\frac{d_0\tau}{d_1}\right)d\tau, \tag{20}$$

where

$$\operatorname{sgn}(h)=\begin{cases}1, & \forall h>0\\ 0, & \forall h=0;\\ -1, & \forall h<0\end{cases}\quad L_1(2\sqrt{|h|\xi})=\begin{cases}J_1(2\sqrt{|h|\xi}), & \forall h\geqslant 0,\\ I_1(2\sqrt{|h|\xi}), & \forall h<0.\end{cases} \tag{20a}$$

We use (3) to obtain the inverse of the transform $U_1[K(p)]$ where $K_1(p)$ is a rational function which can be expanded into partial fractions of the form

$$K(p)=d_1p+d_0+\sum_{m=1}^{n}\frac{h_m}{p+p_m},\quad p_{m+1}>p_m>0. \tag{21}$$

Let $r_1[n](t,\tau)$ be the inverse transform

$$r_1^{[m]}(t,\tau)=L^{-1}\exp\left(-\frac{h_m\tau}{p+p_m}\right)=\delta(t)-\exp(-p_mt)\sqrt{\frac{|h_m|\tau}{t}}\operatorname{sgn}(h_m)L_1(2\sqrt{|h_m|\tau t}), \tag{21a}$$

where $\operatorname{sgn}(h_m)$ and $L_1(2\sqrt{|h_m|\tau t})$ have the same meaning as in (20a). Using the convolution theorem we can write a recurrence relation

$$r_n(t,\tau)=L^{-1}\exp\left(-\tau\sum_{m=1}^{n}\frac{h_m}{p+p_m}\right)=\int_0^t r_1^{[n]}(\xi,\tau)\,r_{n-1}(t-\xi,\tau)\,d\xi=r_{n-1}(t,\tau)- \tag{21b}$$

$$-\operatorname{sgn}(h_n)\sqrt{|h_n|\tau}\int_0^t\exp(-p_n\xi)\,L_1(2\sqrt{|h_n|\xi\tau})\,r_{n-1}(t-\xi,\tau)\frac{d\xi}{\sqrt{\xi}},$$

where $r_1(t,\tau)=r_1^{[1]}(t,\tau)$. Finally

$$k(t,\tau)=H(t-d_1\tau)\exp(-d_0\tau)\,r_n(t-d_1\tau,\tau), \tag{21c}$$

and the inverse transform takes the form

$$u(t)=L^{-1}U_1\left(d_1p+d_0+\sum_{m=1}^{n}\frac{h_m}{p+p_m}\right)=\int_0^{t/d_1}u_1(\tau)\exp(-d_0\tau)\,r_n(t-d_1\tau,\tau)\,d\tau.$$

Equation (21) describes the class of rational functions with finite n and can be extended to the class of meromorphic functions where $n\to\infty$. With the help of (21) and (21a) we can obtain either exact inverses or approximate expressions with an estimate of the error.

Various combinations of (2), (3), and (4) through (7) can be used very effectively in finding inverse transforms. We show this with an example. It follows from (1) and (1a) that multiplication of $K(p)$ by a constant $b>0$ changes the scale of variable τ in the function $k(t,\tau)$:

$$u(t)=L^{-1}U_1[bK(p)]=\int_0^\infty u_1(\tau)\,k(t,b\tau)\,d\tau. \tag{22}$$

This relation is a generalization of the similarity property to which it reduces when $K(p)=p$. Using (22), a large class of rational functions $K(p)$ can be expanded into finite continued fractions [5] of the form

$$K(p)=p+p_1+\frac{h_1}{p+p_2}+\frac{h_2}{p+p_3}+\dots+\frac{h_{n-1}}{p+p_n}. \tag{23}$$

If we let $K_m(p)$ be the function

$$K_m(p) = p + p_m + \frac{h_m}{p} = K_m^0(p) + K_m^1(p), \tag{24}$$

where $K_m^0 = p + p_m$ and $K_m^1(p) = h_m/p$ are its linear and nonlinear parts, then K(p) can be written as a superposition of functions:

$$K(p) = K_1^0 + K_1^1[K_2^0 + K_2^1[\dots[K_{n-1}^0 + K_{n-1}^1[K_n^0(p)]]\dots]. \tag{24a}$$

Since $k_n^{[m]}(t, \tau) \equiv L^{-1}\exp[-\tau K_n^{[m]}(p)]$ and using $r_1^{[m]}(t, \tau)$ and (3), (4) we obtain a recurrence relation for use in calculating the function $k(t, \tau) = k_n^{[1]}(t, \tau)$:

$$k_n^{[n]}(t, \tau) = L^{-1}\exp[-K_n^{[0]}(p)\tau] = \exp(-p_n\tau)\delta(t-\tau);$$

$$k_n^{[n-1]}(t, \tau) = L^{-1}\exp\{-\tau[K_{n-1}^0 + K_{n-1}^1[K_n^{[n]}(p)]]\} =$$

$$= \exp(-p_{n-1}\tau - p_n t)[\delta(t-\tau) - \mathrm{H}(t-\tau)\,\mathrm{sgn}(h_{n-1})\sqrt{\frac{|h_{n-1}|\tau}{t}}L_1(2\sqrt{|h_{n-1}|\tau t})];$$

$$k_n^{[m]}(t, \tau) = L^{-1}\exp\{-\tau[K_m^0 + K_m^1[K_n^{[m+1]}(p)]]\} = \tag{25}$$

$$= \mathrm{H}(t-\tau)\exp(-p_m\tau)[k_n^{[m+1]}(t-\tau, 0) - \mathrm{sgn}(h_m)\int_0^\infty \sqrt{\frac{|h_m|\tau}{\xi}}L_1(2\sqrt{|h_m|\tau\xi})\,k_n^{[m+1]}(t-\tau, \xi)\,d\xi],$$

$$m = n-1, \dots, 2, 1.$$

For example when n = 3 we obtain from (25)

$$K(p) = p + p_1 + \frac{h_1}{p+p_2} + \frac{h_2}{p+p_3} = p + p_1 + \frac{h_1(p+p_3)}{p^2 + (p_2+p_3)p + h_2} \tag{26}$$

and

$$k(t, \tau) = k_3^{[1]}(t, \tau) = \exp(-p_1\tau)[\exp(-(p_2+p_3)t)\delta(t-\tau) -$$

$$- \mathrm{H}(t-\tau)\exp(-p_2\tau - p_3 t)\,\mathrm{sgn}(h_1)\sqrt{\frac{|h_1|\tau}{t}}L_1(2\sqrt{|h_1|\tau(t-\tau)}) +$$

$$+ \mathrm{H}(t-\tau)\,\mathrm{sgn}(h_1 h_2)\sqrt{\frac{|h_1 h_2|\tau}{t-\tau}}\exp(-p_3(t-\tau))\int_0^{t-\tau}\exp(-p_2\xi)\,L_1(2\sqrt{|h_2|\xi(t-\tau)})\,L_1(2\sqrt{|h_1|\tau\xi})\,d\xi].$$

Equations (23)-(26) can be used to obtain exact and approximate expressions for inverse transforms. With the help of the results given here, inverse transforms can effectively be obtained for various other classes of transforms, including more general forms. For example, repeated application of (4) to (21) and (23)-(26) with the use of (19) gives an expression for k(t, τ) corresponding to the class of transforms $K(p) = c_1 p + c_0 + \sqrt{K_1(p)}$, where $K_1(p)$ can be a rational function expanded in partial fractions [Eq. (21)] or a finite continued fraction [Eq. (23].

Transforms like (21) and (23) occur in heat-conduction problems for materials with a continuous thermal memory of the Fourier type [1, 6] and describe many of these materials, enough for practical application. In a Maxwell type material [1, 6-8] the corresponding transform has the form $K_0(p) = \sqrt{K(p)}$, where a rather wide class of functions K(p) can be described with rational functions of the form $K(p) = d_2p^2 + d_1p + d_0 + R_n(p)/R_{n+1}(p)$, where $R_n(p)$ is a polynomial of degree n in p. Letting $K_0(p)$ be a finite continued fraction (23), we can approximate $\sqrt{K(p)}$ to order p^{-m}. Examining (25), we note that k(t, τ) corresponding to (23) can be written in the form $k(t, \tau) = H(t-\tau)k_0(t-\tau, t, \tau)$, where the difference of arguments t − τ appears as a single unit. This argument is independent of time and corresponds to heat propagation with a finite velocity. Thus we deduce the important result that the heat propagation velocity in an arbitrary Maxwell medium does not change with time.

With the help of a change of variable in the Laplace transform a connection can be established between solutions of equations differing from each other by time operators. For example in [1, 6] a connection was established using (1) between the solutions of similar heat-conduction problems in materials with and without memory. A correspondence principle between the transforms of such similar problems was obtained. In [7, 8], using (10), the solutions of heat-conduction boundary-value problems were obtained in materials with memory. Obviously the above results can also be used in the solution of other problems in heat transfer, mass transfer, momentum transfer in compound materials such as composites and materials with memory, and also in the solution of various types of coupled heat and mass transfer problems.

NOTATION

L, L^{-1}, direct and inverse Laplace transform operators; p, Laplace transform variable; $U(t)$, $U(p)$, original function and its transform; $H(t)$, Heaviside unit step function; $\delta(t)$, Dirac delta function; $\Gamma(x)$, Gamma function; J_1, I_1, Bessel functions of the first kind and first order for real and imaginary arguments, respectively.

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